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Oscillations of charged particles in an external magnetic field about steady motion

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Abstract. We develop a Hamiltonian formalism that can be used to study the particle dynamics near stable equilibria. The construction of an original canonical transformation allowed us to prove the conservation of the linear momentum P_3, which permitted the expansion of the Hamiltonian about a fixed point. The definition of the rotational variable h whose Poisson algebra properties played the essential role in the diagonalization of the quadratic Hamiltonian yielding two uncoupled oscillators with definite frequencies and amplitudes. It is through applying this variable near a fixed point that come to light Heisenberg's and Harmonic Oscillator equations of motion of the particles, leading thus the association of the fixed point trajectories with arbitrary trajectories in its immediate neighborhood. The present formalism succeeded to treat the problem of free-electron laser dynamics and may be applied to similar cases.

PACS. 52.30.-q Plasma dynamics and flow – 52.30.Cv Magnetohydrodynamics (including electron magnetohydrodynamics) – 52.30.Gz Gyrokinetics

1 Introduction

The dynamics of charged particles in electric and magnetic fields is of both academic and practical interest. The areas where this problem finds applications include the development of accelerator physics [1], plasma physics [2], nuclear physics [3], free-electron lasers [4] and so on.

In this paper, we develop a Hamiltonian formalism that proved to be efficient in treating the problem of particle dynamics in a free-electron laser consisting of a helical wiggler magnetic field and a uniform guide field [5,6]. It is through the construction of the canonical transformation of Section 2 and the definition of a rotational variable h that our work was made suitable to deal with the physical insight of the problem. As a matter of fact, we restrict ourselves to two rotational variables $h_1 = \sqrt{P_1}\exp(\mathrm{i}Q_1)$ and $h_2 = \sqrt{P_2}\exp(\mathrm{i}Q_2)$ corresponding to the gyroradius motion and the guiding center motion, respectively. We draw attention that our canonical transformation in its present form is expressed in terms of real and imaginary parts of these *rotational* variables. Moreover, the canonical transformation allowed us to find out the constant of motion P_3. Both *Heisenberg's* picture of motion and simple harmonic oscillator equation are found by applying $\hat{h}$ to the case of a fixed point and the resulting Poisson algebra yields two uncoupled harmonic oscillators. Although not exact, the new physical system is integrable allowing then a satisfactory description of the trajectories within the immediate neighborhood of fixed points (ideal helical trajectories). The investigation of the oscillator characteristic frequencies $\Omega_\pm$ leads to the complete solution to the problem in the quadratic approximation; allowing one to study the different modes of propagation and to identify, and then avoid the problematic operating conditions of the concerned system. The present formalism can be applied to the case of particles under the effect of a transverse magnetic field as those encountered in Helmholtz coils or in NMR experiments [7–9].

The organization of the paper is as follows. The canonical transformation and the constants of motion are given in Section 2. Section 3 is devoted to the rotational variable and Hamiltonian approximation near stable equilibria finding out the characteristic frequencies giving thereafter the complete solution of the problem. We end by the conclusion. The construction of the canonical transformation is given in the Appendix.

2 Canonical transformation and constants of motion

As stated in the introduction, the Hamiltonian that will be treated is that of an electron in a free-electron laser consisting of a helical wiggler magnetic field and a uniform

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guide field [5,6]. After having performed three canonical transformations (see the Appendix), the resulting transformation $(q_1, q_2, q_3, p_1, p_2, p_3) \mapsto (Q_1, Q_2, Q_3, P_1, P_2, P_3)$ is given by:

$$q_1 = \gamma \left[\sqrt{P_1}\cos(Q_3+Q_1) - \sqrt{P_2}\sin(Q_3-Q_2)\right] \quad (1)$$

$$q_2 = \gamma \left[\sqrt{P_1}\sin(Q_3+Q_1) + \sqrt{P_2}\cos(Q_3-Q_2)\right] \quad (2)$$

$$q_3 = Q_3 \quad (3)$$

$$p_1 = \frac{1}{\gamma}\left[-\sqrt{P_1}\sin(Q_3+Q_1) + \sqrt{P_2}\cos(Q_3-Q_2)\right] \quad (4)$$

$$p_2 = \frac{1}{\gamma}\left[\sqrt{P_1}\cos(Q_3+Q_1) + \sqrt{P_2}\sin(Q_3-Q_2)\right] \quad (5)$$

$$p_3 = P_3 - L_3 \quad (6)$$

where all parameters and variables are dimensionless. The canonical angular momentum L_3 is given by:

$$L_3 = q_1 p_2 - q_2 p_1 = P_1 - P_2. \quad (7)$$

Since the electron is expected to rotate about the q_3-axis, one can state that the generating function corresponding to an infinitesimal rotation about the same axis is,

$$G = \mathbf{L} \cdot \hat{\mathbf{q}}_3 = L_3. \quad (8)$$

With the help of equation (11), the last equation of the transformation gives

$$p_3 = P_3 - G = P_3 - L_3. \quad (9)$$

It should be noted that the canonical angular momentum as defined here is different from the mechanical angular momentum. This is due to the fact that the forces on the system are of velocity-dependent potential type [10].

In order to prove that P_3 is a constant of motion, we evaluate the following Poisson brackets:

$$\begin{aligned} [Q_1, L_3] &= [Q_2, p_3] = 1 \\ [Q_2, L_3] &= [Q_1, p_3] = -1 \end{aligned} \quad (10)$$

which gives:

$$p_3 + L_3 = \mathit{constant} \quad (11)$$

yielding the conserved quantity:

$$P_3 = \mathit{constant}. \quad (12)$$

As a matter of fact, the transformation of Chen and Davidson [11] who have studied the problem of the electron dynamics in a free electron laser is indeed canonical but it violates equation (7). Their canonical transformation gives:

$$L_z = xP_y - yP_x \neq P_\phi - P_\Psi. \quad (13)$$

This violation is due to the missing sine terms in the x and y components of their transverse momenta (Eqs. (14, 15) of Ref. [11]). As a consequence, the transformation of Chen and Davidson succeeded to treat the one dimensional (1D) case of FELs where the inevitable dependence of the wiggler on the transverse spatial variables is neglected, meanwhile it proved to be non compatible with the three-dimensional case.

3 Rotational variable and Hamiltonian formalism

Defining a new canonical variable (rotational variable) as:

$$h\left(Q_h = \Omega_h t,\ P_h\right) = \sqrt{P_h}\exp\left(\mathrm{i}Q_h\right) \quad (14)$$

where Q_h is an angular generalized co-ordinate and P_h is the corresponding conjugate momentum.

From the definition (14), we obtain the following Poisson brackets identities

$$[h, h^*] = \mathrm{i} \quad (15)$$

$$[h, H] = \dot{h} = \mathrm{i}h\dot{Q}_h + \frac{\dot{P}_h}{2h^*} \quad (16)$$

$$[[h, H], H] = \ddot{h} = \frac{1}{h^*}\left(\frac{-\dot{P}_h^2}{4P_h} + \mathrm{i}\dot{Q}_h\dot{P}_h + \frac{\ddot{P}_h}{2}\right) + h\left(\mathrm{i}\ddot{Q}_h - \dot{Q}_h^2\right) \quad (17)$$

where the point and star denote time derivative and complex conjugate respectively, and H is the Hamiltonian. It is clear from the above equations that the conjugate momentum of the rotational variable h is $(-\mathrm{i}h^*)$.

Referring to the canonical transformation (1–6), we obtain two specific rotational variables corresponding to the transverse coordinates and momenta.

$$\begin{aligned} h_1 &= \sqrt{P_1}\exp(\mathrm{i}Q_1) \\ &= \frac{\gamma}{2}\left[\frac{(q_1 + \mathrm{i}q_2)}{\gamma^2} - \mathrm{i}\left(p_1 + \mathrm{i}p_2\right)\right]\exp\left(-\mathrm{i}Q_3\right), \end{aligned} \quad (18)$$

$$\begin{aligned} h_2 &= \sqrt{P_2}\exp(\mathrm{i}Q_2) \\ &= \frac{\mathrm{i}\gamma}{2}\left[\frac{(q_1 - \mathrm{i}q_2)}{\gamma^2} - \mathrm{i}\left(p_1 - \mathrm{i}p_2\right)\right]\exp\left(\mathrm{i}Q_3\right). \end{aligned} \quad (19)$$

For a fixed point $\left(\dot{P}_h = 0 \Rightarrow P_h = P_0 = \mathit{constant}\right)$, equation (16) is reduced to the *Heisenberg's equation* where the states do not depend on time but the physical quantities change:

$$h\left(Q_h = \Omega_h t,\ P_0\right) = \sqrt{P_0}\exp\left(\mathrm{i}Q_h\right) \quad (20)$$

while equation (17) leads to the well-known simple harmonic oscillator motion:

$$[h, H] = \dot{h} = \mathrm{i}\Omega_h h \quad (21)$$

$$[[h, H], H] = \ddot{h} = -\Omega_h^2 h \quad (22)$$

which gives,

$$[hh^*, H] = 0 \quad (23)$$

and,

$$H = hh^*\Omega_h + \mathit{constant}. \quad (24)$$

If the constant of motion P_3 is such that a fixed point of the Hamiltonian exists, we expand the Hamiltonian about the fixed point of coordinates:

$$h_{10} = \sqrt{P_{10}}\exp\left(\mathrm{i}Q_{10}\right), \qquad h_{20} = \sqrt{P_{20}}\exp\left(\mathrm{i}Q_{20}\right),$$
$$h_{10}^* = \sqrt{P_{10}}\exp\left(-\mathrm{i}Q_{10}\right), \qquad h_{20}^* = \sqrt{P_{20}}\exp\left(-\mathrm{i}Q_{20}\right).$$

The deviations of the rotationals from equilibrium are denoted by:

$$h_i = h_{i0} + \eta_i. \tag{25}$$

The Hamiltonian up to the quadratic term is then:

$$H(P_3, h_1, h_2) = H_0(P_3, h_{10}, h_{20}) + \Delta H \tag{26}$$

where the quadratic part is:

$$\Delta H = \frac{1}{2}\sum_{i,j=1}^{4}\left(\frac{\partial^2 H}{\partial h_i \partial h_j}\right)_0 \eta_i \eta_j \tag{27}$$

and H_0 is the fixed part of the Hamiltonian

$$h_3 \equiv h_1^*, \quad \eta_3 \equiv \eta_1^*, \quad h_4 \equiv h_2^*, \quad \eta_4 \equiv \eta_2^*. \tag{28}$$

In order to determine the characteristic frequencies of ΔH, we rewrite the rotational of equation (14) as a linear combination of the different deviations from the fixed point:

$$h = \alpha_1\eta_1 + \alpha_2\eta_2 + \alpha_3\eta_3 + \alpha_4\eta_4 \tag{29}$$

where the α_i's are unknown coefficients.

Referring to equation (21), the Poisson bracket of h and ΔH is then:

$$\begin{aligned}[h, \Delta H] &= \sum_{k=1}^{2} \mathrm{i}\left\{\frac{\partial h}{\partial \eta_k}\frac{\partial \Delta H}{\partial \eta_{(k+2)}} - \frac{\partial \Delta H}{\partial \eta_k}\frac{\partial h}{\partial \eta_{(k+2)}}\right\} \\ &= \mathrm{i}\Omega_h h \end{aligned} \tag{30}$$

depending explicitly on ΔH. This leads to the characteristic polynomial:

$$\Omega^4 + b\Omega^2 + c = 0 \tag{31}$$

where b and c are constant coefficients.

The squared characteristic frequencies roots of equation (31) are then given by:

$$\Omega_\pm^2 = \frac{-b \pm \left(b^2 - 4c\right)^{1/2}}{2}\cdot \tag{32}$$

The squared frequencies are then determined and we may solve for the coefficients α_i.

As a matter of fact, the normalization of the *rotational* variable ensured by equation (30) is equivalent to the following relationship between the α_i:

$$[h, h^*] = \mathrm{i}\left(\alpha_1^2 + \alpha_2^2 - \alpha_3^2 - \alpha_4^2\right) \tag{33}$$

imposing then the following condition:

$$\alpha_1^2 + \alpha_2^2 - \alpha_3^2 - \alpha_4^2 = 1. \tag{34}$$

This normalization condition not only fixes the coefficients α_i, but also determines the sign of the frequency Ω.

Referring to equation (24), the Hamiltonian may be written as:

$$H = H_0 + \Delta H = H_0 + h_+h_+^*\Omega_+ + h_-h_-^*\Omega_-. \tag{35}$$

Thus, the right hand side of equation (35) gives the *rotational* variables, which are solutions to Hamilton's equations defined by:

$$h_\pm(t) = h_\pm(t=0)\,\mathrm{e}^{\mathrm{i}\Omega_\pm t} = \sqrt{P_\pm(t=0)}\,\mathrm{e}^{\mathrm{i}\Omega_\pm t} \tag{36}$$

giving thus two obvious constants of motion $\hat{P}_+(0)$ and $\hat{P}_-(0)$.

The complete solution to the problem, in the present approximation, has thus been found. It is worthy to note that expressed in terms of the characteristic solutions, the quadratic Hamiltonian of equation (35) is rewritten as:

$$\begin{aligned}\Delta H &= \Omega_+h_+(0)h_+^*(0) + \Omega_-h_-(0)h_-^*(0) \\ &= \Omega_+P_+(0) + \Omega_-P_-(0)\end{aligned} \tag{37}$$

which shows a system of two uncoupled harmonic oscillators (Q_+, P_+) and (Q_-, P_-). In fact, the quantities $h_\pm$ defined by equation (36) are redefined in the perturbed system where their time dependence is determined by the equation of motion (30) and one may write the rigorous equations of motion corresponding to the Hamiltonian (37) as:

$$\dot{h}_\pm = \left[h_\pm, h_\pm^*\right]\frac{\partial \Delta H}{\partial h_\pm^*}\cdot \tag{38}$$

Constructing new generalized coordinates $\Theta_\pm$ and conjugate momenta $\Pi_\pm$ as:

$$\Theta_\pm = \sqrt{2P_\pm}\sin(Q_\pm), \quad \Pi_\pm = \sqrt{2P_\pm}\cos(Q_\pm)$$

equation (37) takes the form:

$$\Delta H = \frac{1}{2}\Omega_+\left(\Pi_\pm^2 + \Theta_+^2\right) + \frac{1}{2}\Omega_-\left(\Pi_-^2 + \Theta_-^2\right) \tag{39}$$

which proves that ΔH admits two different forms: one is a polynomial of degree four in the symplectic phase variables $\Pi_\pm$ and $\Theta_\pm$ while the other is a polynomial of degree two in the variables $P_\pm = \left(\Pi_\pm^2 + \Theta_\pm^2\right)/2$. It is clear that the trajectories of this system are conditionally-periodic windings on the surfaces $P_\pm = constant$ with frequencies $\partial\Delta H/\partial P_\pm$.

4 Conclusion

In this paper, we develop a Hamiltonian formalism that proved to be efficient in treating the problem of the free electron laser (FEL) dynamics [5,6]. It is through the construction of the canonical transformation of Section 2 and the definition of a rotational variable h that our work was made suitable to deal with the physical insight of the problem. As a matter of fact, we restrict ourselves to two rotational variables $h_1 = \sqrt{P_1}\exp(\mathrm{i}Q_1)$ and $h_2 = \sqrt{P_2}\exp(\mathrm{i}Q_2)$ corresponding to the gyroradius motion and the guiding center motion, respectively. We draw attention that our canonical transformation in its present form is expressed in terms of real and imaginary parts of these *rotational* variables. Moreover, the canonical transformation allowed us to find out the constant of motion P_3. Both *Heisenberg's* picture of motion and simple harmonic oscillator equation are found by applying $\hat{h}$ to the case of a fixed point and the resulting Poisson algebra yields two uncoupled harmonic oscillators. Although not exact, the new physical system is integrable allowing then a satisfactory description of the trajectories within the immediate neighborhood of fixed points (ideal helical trajectories). The investigation of the oscillator characteristic frequencies $\Omega_\pm$ leads to the complete solution to the problem in the quadratic approximation; allowing one to study the different modes of propagation and to identify, and then avoid the problematic operating conditions of the concerned system. The present formalism can be applied to the case of particles under the effect of a transverse magnetic field as those encountered in Helmholtz coils or in NMR experiments [7–9].

Appendix

To verify that the transformation given in equations (1–6) is canonical; we first perform the following canonical transformation among the space coordinates and momenta,

$$q_1 = \gamma\left(P_a - \mathrm{i}Q_b\right), \quad q_2 = \mathrm{i}\gamma\left(P_b - \mathrm{i}Q_a\right), \quad q_3 = Q_c,$$
$$p_1 = \frac{\mathrm{i}}{2\gamma}\left(P_b + \mathrm{i}Q_a\right), \; p_2 = \frac{1}{2\gamma}\left(P_a + \mathrm{i}Q_b\right), \; p_3 = P_c, \tag{A.1}$$

given by the generating function

$$F_3(p_1, p_2, p_3; Q_a, Q_b, Q_c) = -(2\gamma p_2 - \mathrm{i}Q_b)(\gamma p_1 + Q_a) + \mathrm{i}\gamma p_1 Q_b - p_3 Q_c. \tag{A.2}$$

Second, we introduce the polar coordinates α, β, P_α and P_β in the transverse phase plane,

$$Q_a = \sqrt{2P_\alpha}\sin(\alpha), \;\; Q_b = \sqrt{2P_\beta}\sin(\beta), \;\; Q_c = \delta$$
$$P_\alpha = \sqrt{2P_\alpha}\cos(\alpha), \;\; P_b = \sqrt{2P_\beta}\cos(\beta), \;\; P_c = P_\delta \tag{A.3}$$

with the generating function

$$F_3\left(P_a, P_b, P_c; \alpha, \beta, \delta\right) = -\frac{1}{2}P_a^2\tan(\alpha) + \frac{1}{2}P_b^2\tan(\beta) - \delta P_c. \tag{A.4}$$

Finally, the canonical transformation,

$$\alpha = Q_1 + Q_3, \quad \beta = Q_2 - Q_3, \quad \delta = Q_3,$$
$$P_\alpha = P_1, \qquad P_\beta = P_2, \qquad P_\delta = P_3 - P_1 + P_2, \tag{A.5}$$

with the generating function

$$F_3(P_\alpha, P_\beta, P_\delta; Q_1, Q_2, Q_3) = -(Q_1 + Q_3)P_\alpha - (Q_2 - Q_3)P_\beta - Q_3 P_\delta \tag{A.6}$$

yields the resulting canonical transformation in equations (1–6).

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